

# Multiple (inverse) binomial sums of arbitrary weight and depth and the all-order $\varepsilon$ -expansion of generalized hypergeometric functions with one half-integer value of parameter

Mikhail Yu. Kalmykov,<sup>ab\*</sup> Bennie F.L. Ward<sup>a</sup> and Scott A. Yost<sup>a†</sup>

<sup>a</sup>*Department of Physics, Baylor University,  
One Bear Place, Box 97316, Waco, TX 76798-7316, U.S.A.*

<sup>b</sup>*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,  
141980 Dubna (Moscow Region), Russia*

*E-mail: kalmykov@theor.jinr.ru, BFL\_Ward@baylor.edu, syost@princeton.edu*

ABSTRACT: We continue the study of the construction of analytical coefficients of the  $\varepsilon$ -expansion of hypergeometric functions and their connection with Feynman diagrams. In this paper, we show the following results:

Theorem A: The multiple (inverse) binomial sums

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}^k} \frac{z^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1),$$

where  $k = \pm 1$ ,  $S_a(j)$  is a harmonic series,  $S_a(j) = \sum_{k=1}^j \frac{1}{k^a}$ , and  $c$  is any integer number are expressible in terms of Remiddi-Vermaseren functions;

Theorem B: The hypergeometric functions

$${}_pF_{p-1}\left(\vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon, \frac{1}{2} + B_{p-1}; z\right), \quad {}_pF_{p-1}\left(\vec{A} + \vec{a}\varepsilon, \frac{1}{2} + A_p; \vec{B} + \vec{b}\varepsilon; z\right),$$

are expressible in terms of the harmonic polylogarithms of Remiddi and Vermaseren with coefficients that are ratios of polynomials.

KEYWORDS: NLO Computations, Integrable Hierarchies.

\*Present Address: II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany.

†Present address: Department of Physics, Princeton University, Princeton, NJ 08540, U.S.A.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Analytical evaluation of a basis of multiple (inverse) binomial sums of arbitrary weight and depth</b>	<b>4</b>
2.1 Some analytical properties of multiple (inverse) binomial sums of arbitrary weight and depth	4
2.2 Analytical evaluation of multiple (inverse) binomial sums of arbitrary weight and depth	6
<b>3. All-order <math>\varepsilon</math>-expansion of hypergeometric functions with one half-integer value of the parameters via multiple (inverse) binomial sums</b>	<b>13</b>
<b>4. Generalized multiple (inverse) binomial sums via derivatives of generalized hypergeometric functions</b>	<b>14</b>
<b>5. Discussion and conclusions</b>	<b>17</b>
<b>A. Zoo of special functions</b>	<b>19</b>

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## 1. Introduction

Feynman diagrams [1] are a primary tool for calculating radiative corrections to any processes within the Standard Model or its extensions. With increasing of accuracy of measurements, more and more complicated diagrams (with increasing number of loops and legs and increasing numbers of variables associated with different particle masses) must be evaluated. The essential progress in such calculations is often associated with the invention of new (mainly mathematical) algorithms (e.g. refs. [2, 3]) and their realization as a computer programs (e.g. refs. [4, 5]). One fruitful approach to the calculation of Feynman diagrams is based on their representation in terms of hypergeometric functions [6] or multiple series [7, 8]. We will refer to such representations as *hypergeometric representations* for Feynman diagrams. Unfortunately, there does not exist a universal hypergeometric representation for all types of diagrams. Constructing these representations is still a matter of the personal experience of the researcher [9–13]. Nevertheless, existing experience with Feynman diagrams leads us to expect that all Feynman diagrams should be associated with hypergeometric functions.

For practical applications, finding a hypergeometric representation is not enough. It is necessary to construct the so-called  $\varepsilon$ -expansion, which we may understand as the construction of the analytical coefficients of the Laurent expansion of hypergeometric functions around rational values of their parameters. In this direction, very limited results are

available.<sup>1</sup> The pioneering systematic activity in studying the Laurent series expansion of hypergeometric functions at particular values of the argument ( $z = 1$ ) was started by David Broadhurst [14] in the context of Euler-Zagier sums (or multidimensional zeta values) [15]. This activity has received further consideration for another, physically interesting point,  $z = 1/4$  (see the relevant appendix in ref. [8, 10]), and also for the “primitive sixth roots of unity” (see ref. [16]). Over time, other types of sums<sup>2</sup> have been analysed in a several publications: *harmonic* sums [7, 17], *generalized harmonic sums* [18, 12], *binomial* sums [11, 12] and *inverse binomial* sums [12, 19].

The introduction of new functions, such as *multiple polylogarithms*, (see appendix A) independently in mathematics and physics [16, 20–23],<sup>3</sup> allows us to derive a set of universal algorithms for the simplification and construction of the analytical coefficients of the Laurent expansion of a large class of hypergeometric functions. (For details, see refs. [10, 13, 17, 18, 26–29].) Recently, similar problems have also drawn the attention of mathematicians [28, 30]. However, the general solution of this problem remains unknown.

The multiple series representation has further applications in the framework of Feynman diagram calculations. In particular, the Smirnov-Tausk approach [2, 3] (see also ref. [31]) was very productive for constructing the analytical coefficients of the  $\varepsilon$ -expansion (finite part mainly) of Feynman diagrams depending on one or two massless (ratio of massive) kinematic variables. Presently, there are several computer realizations of this approach [5]. In the framework of this technique, the Feynman parameter representation [1] for a diagram is rewritten in terms of multiple Mellin-Barnes (contour integral) representations, resulting in expressions for which a Laurent expansion about  $\varepsilon = 0$  may be constructed explicitly, using gamma functions and their derivatives. The results may be summed analytically or numerically, typically leading to the same sums as in the construction of the  $\varepsilon$ -expansion of hypergeometric functions: the (*generalized*) *harmonic sums* and (*inverse*) *binomial sums*. *Inverse binomial sums* typically arise from massive loops; see refs. [6, 32]. Another source of multiple sums in Feynman diagrams comes from the Frobenius series solution of a differential equation [33]. Other classes of sums have been considered as well.<sup>4</sup>

Analytical results are possible when these sums can be evaluated explicitly. For the analysis of (*generalized*) *harmonic sums*, the *nested sums approach* [17, 18] permits the reduction of any type of (*generalized*) *harmonic sum* to a set of basis sums. The analytical evaluation of these basis sums is an independent problem. (See, for example, ref. [12].)

<sup>1</sup>One of the *classical tasks* in mathematics is to find the full set of parameters and arguments for which hypergeometric functions are expressible in terms of algebraic functions. Quantum field theory makes a *quantum* generalisation of this classical task: to find the full set of parameters and arguments so that the all-order  $\varepsilon$ -expansion is expressible in terms of known special functions or identify the full set of functions which must be invented in order to construct the all-order  $\varepsilon$ -expansion of generalized hypergeometric functions.

<sup>2</sup>See eq. (2.1) for clarifying of terminology.

<sup>3</sup>Hyperlogarithms have been considered by Kummer, Poincaré, and Lappo-Danilevsky; see [24]. The interrelation between hyperlogarithms and multiple polylogarithms has been discussed in [25].

<sup>4</sup>*Finite harmonic sums* are another class, on which more details may be found in ref. [34]. However, there presently does not exist an appropriate generalization of *multiple (inverse) binomial sums* to finite harmonic sums. Some recent attempts in this direction have been discussed in refs. [18, 35].

The *Generating function approach* [36] is a universal method for analytically evaluating arbitrary sums, which was successfully applied (see section (2.3) in ref. [12]) for an analysis of multiple (inverse) binomial sums [11, 12]. The generating function approach allows us to convert arbitrary sums to a system of differential equations. The question of the expressibility of the solution to this differential equation in terms of known (special) functions is not addressed by this approach. In particular, the partial results of refs. [7, 11–13] were restricted by attempts to express the results of the calculation in terms of only classical or Nielsen polylogarithms [37, 38]. It is presently unknown what type of sums (beyond generalized harmonic sums) are expressible in terms of known special functions.<sup>5</sup>

The aim of this paper is to prove the following theorems:

**Theorem A.** *The multiple (inverse) binomial sums*

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}^k} \frac{z^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1), \quad (1.1)$$

where  $k = \pm 1$ ,  $S_a(j) = \sum_{k=1}^j \frac{1}{k^a}$  is a harmonic series, and  $c$  is any integer, are expressible in terms of Remiddi-Vermaseren functions with

- (1) for  $k = 1$  : (i)  $c \geq 2$  rational coefficients; (ii)  $c \leq 1$  ratios of polynomials;
- (2) for  $k = -1$  : (i)  $c \geq 1$  rational coefficients; (ii)  $c \leq 0$  ratios of polynomials.

**Theorem B.** *The all order  $\varepsilon$ -expansion of the generalized hypergeometric functions* [41]

$${}_pF_{p-1} \left( \vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon, \frac{1}{2} + I_1; z \right), \quad (1.2a)$$

$${}_pF_{p-1} \left( \vec{A} + \vec{a}\varepsilon, \frac{1}{2} + I_2; \vec{B} + \vec{b}\varepsilon; z \right), \quad (1.2b)$$

where  $\vec{A}, \vec{B}$  are lists of integers and  $I_1, I_2$  are integers, are expressible in terms of the harmonic polylogarithms with coefficients that are ratios of polynomials.

The paper is organised as follows. In section 2, we will prove theorem A. In section 3, the results of theorem A will be applied to hypergeometric functions to prove theorem B. Section 4 is devoted to a discussion of an algorithm for the reduction and analytical evaluation of generalized multiple (inverse) binomial sums. Appendix A contains some basic information about relevant special functions.

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<sup>5</sup>There is not universal agreement on what it means to express a solution in terms of known special functions. One reasonable answer has been presented by Kitaev in the Introduction to ref. [39]. where he quotes R. Askye’s Forward to the book *Symmetries and Separation of Variables* by W. Miller, Jr., [40] which says “One term which has not been defined so far is ‘special function’. My definition is simple, but not time invariant. A function is a special function if it occurs often enough so that it gets a name”. Kitaev adds, “. . . most of the people who apply them . . . understand, under the notion of special functions, a set of functions which can be found in one of the well-known reference books. . . .” To this, we may add “functions which can be found in one of the well-known computer algebra systems.”

## 2. Analytical evaluation of a basis of multiple (inverse) binomial sums of arbitrary weight and depth

The main purpose of this section is to prove theorem A. In the first subsection, we will consider differential equations satisfied by multiple (inverse) binomial sums, and use the analytical properties of such sums to derive two useful lemmas. In the second subsection, we prove auxiliary propositions for the separate cases of multiple binomial sums and inverse binomial sums, and use them to complete the proof of theorem A.

### 2.1 Some analytical properties of multiple (inverse) binomial sums of arbitrary weight and depth

Let us define the *multiple sums*

$$\Sigma_{a_1, \dots, a_p; b_1, \dots, b_q; c}^{(k)}(u) \equiv \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}^k} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1) S_{b_1}(2j-1) \cdots S_{b_q}(2j-1), \quad (2.1)$$

where  $S_a(j) = \sum_{k=1}^j \frac{1}{k^a}$  is a harmonic series and  $c$  is any integer. For particular values of  $k$ , the sums (2.1) are called

$$k = \left\{ \begin{array}{l} 0 \text{ generalized harmonic} \\ 1 \text{ inverse binomial} \\ -1 \text{ binomial} \end{array} \right\} \text{ sums .}$$

The case  $\Sigma_{a_1, \dots, a_p; 0, \dots, 0; c}^{(0)}(u)$  is called a *harmonic sum*. The number  $w = c + a_1 + \cdots + a_p + b_1 + \cdots + b_q$  is called the *weight* and  $d = p + q$  is called the *depth*.

The general properties of multiple sums can be derived from their generating functions. Let us rewrite the multiple sum (2.1) in the form  $\Sigma_{\vec{a}; \vec{b}; c}^{(k)}(u) = \sum_{j=1}^{\infty} u^j \eta_{\vec{a}; \vec{b}; c}^{(k)}(j)$ , where  $\vec{a} \equiv (a_1, \dots, a_p)$  and  $\vec{b} \equiv (b_1, \dots, b_q)$  denote the collective lists of indices and  $\eta_{\vec{a}; \vec{b}; c}^{(k)}(j)$  is the coefficient of  $u^j$ . In order to find the differential equation for generating functions of *multiple sums* it is necessary to find a recurrence relation for the coefficients  $\eta_{\vec{a}; \vec{b}; c}^{(k)}(j)$  with respect to the summation index  $j$ . Using the explicit form of  $\eta_{\vec{a}; \vec{b}; c}^{(k)}(j)$ , the recurrence relation for the coefficients can be written in the form<sup>6</sup>

$$[2(2j+1)]^k (j+1)^{c-k} \eta_{\vec{a}; \vec{b}; c}^{(k)}(j+1) = j^c \eta_{\vec{a}; \vec{b}; c}^{(k)}(j) + r_{\vec{a}; \vec{b}}^{(k)}(j), \quad (2.2)$$

where the “remainder”  $r_{\vec{a}; \vec{b}}^{(k)}(j)$  is given by

$$\begin{aligned} \binom{2j}{j}^k r_{\vec{a}; \vec{b}}^{(k)}(j) &= \prod_{r=1}^p [S_{a_r}(j-1) + j^{-a_r}] \times \prod_{l=1}^q [S_{b_l}(2j-1) + (2j)^{-b_l} + (2j+1)^{-b_l}] \\ &\quad - \prod_{r=1}^p \prod_{l=1}^q S_{a_r}(j-1) S_{b_l}(2j-1). \end{aligned} \quad (2.3)$$

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<sup>6</sup>We would like to point out that eq. (2.2) is valid for an arbitrary integer  $k$  and  $c - k \geq 0$ . In the case  $c - k < 0$ , the proper term will be generated in the r.h.s. of the equation.

Multiplying both sides of eq. (2.2) by  $u^j$ , summing from  $j = 1$  to  $\infty$ , and using the fact that any extra power of  $j$  corresponds to the derivative  $u(d/du)$  leads to the following differential equations for the generating functions  $\Sigma_{\vec{a};\vec{b};c}^{(k)}(u)$  (see ref. [19]):

$$\left[ \left( \frac{4}{u} - 1 \right) u \frac{d}{du} - \frac{2}{u} \right] \left( u \frac{d}{du} \right)^{c-1} \Sigma_{\vec{a};\vec{b};c}^{(1)}(u) = \delta_{p,0} + R_{\vec{a};\vec{b}}^{(1)}(u), \quad (2.4a)$$

$$\left( \frac{1}{u} - 1 \right) \left( u \frac{d}{du} \right)^c \Sigma_{\vec{a};\vec{b};c}^{(0)}(u) = \delta_{p,0} + R_{\vec{a};\vec{b}}^{(0)}(u), \quad (2.4b)$$

$$\left[ \left( \frac{1}{u} - 4 \right) u \frac{d}{du} - 2 \right] \left( u \frac{d}{du} \right)^c \Sigma_{\vec{a};\vec{b};c}^{(-1)}(u) = 2\delta_{p,0} + 2 \left( 2u \frac{d}{du} + 1 \right) R_{\vec{a};\vec{b}}^{(-1)}(u), \quad (2.4c)$$

where  $R_{\vec{a};\vec{b}}^{(k)}(u) \equiv \sum_{j=1}^{\infty} u^j r_{\vec{a};\vec{b}}^{(k)}(j)$  and  $\delta_{a,b}$  is the Kronecker  $\delta$ -function. The boundary conditions for any of these sums and their derivatives are

$$\left( u \frac{d}{du} \right)^j \Sigma_{\vec{a};\vec{b};c}(0) = 0, \quad j = 0, 1, 2, \dots \quad (2.5)$$

From the analysis in refs. [11–13, 19], we have deduced that the set of equations for the *generating functions* has a simpler form in terms of a new variable. For *multiple inverse binomial sums*, this variable is defined by

$$y = \frac{\sqrt{u-4} - \sqrt{u}}{\sqrt{u-4} + \sqrt{u}}, \quad u = -\frac{(1-y)^2}{y}, \quad (2.6)$$

and for *multiple binomial sums*, it is defined by

$$\chi = \frac{1 - \sqrt{1-4u}}{1 + \sqrt{1-4u}}, \quad u = \frac{\chi}{(1+\chi)^2}. \quad (2.7)$$

Let us consider the differential equation for *multiple inverse binomial sums* in terms of these new variables. The notation  $\Sigma_{\vec{a};\vec{b};c}^{(k)}(y)[(\chi)]$  will be used for a sum defined by eq. (2.1), where the variable  $u$  is rewritten in terms of variable  $y[\chi]$  defined by eq. (2.6) [(2.7)]:

$$\begin{aligned} \Sigma_{\vec{a};\vec{b};c}^{(1)}(y) &\equiv \Sigma_{\vec{a};\vec{b};c}^{(1)}(u(y)) \equiv \Sigma_{\vec{a};\vec{b};c}^{(1)}(u) \Big|_{u=u(y)}, \\ \Sigma_{\vec{a};\vec{b};c}^{(-1)}(\chi) &\equiv \Sigma_{\vec{a};\vec{b};c}^{(-1)}(u(\chi)) \equiv \Sigma_{\vec{a};\vec{b};c}^{(-1)}(u) \Big|_{u=u(\chi)}. \end{aligned} \quad (2.8)$$

In terms of the variable  $y$ , equation (2.4a) may be split into sum of two equations

$$\left( -\frac{1-y}{1+y} y \frac{d}{dy} \right)^{c-1} \Sigma_{\vec{a};\vec{b};c}^{(1)}(y) = \frac{1-y}{1+y} \sigma_{\vec{a};\vec{b}}^{(1)}(y), \quad (2.9a)$$

$$y \frac{d}{dy} \sigma_{\vec{a};\vec{b}}^{(1)}(y) = \delta_{p,0} + R_{\vec{a};\vec{b}}^{(1)}(y) \quad (2.9b)$$

with boundary condition

$$\Sigma_{\vec{a};\vec{b};c}^{(1)}(1) = 0.$$

Equation (2.9a) could be rewritten as

$$\left(\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-j}\Sigma_{\vec{a};\vec{b};c}^{(1)}(y)=\Sigma_{\vec{a};\vec{b};j}^{(1)}(y), \quad (2.10)$$

or in equivalent form

$$\left(-\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-j-1}\Sigma_{\vec{a};\vec{b};c}^{(1)}(y)=\int_1^y dy\left(\frac{2}{1-y}-\frac{1}{y}\right)\Sigma_{\vec{a};\vec{b};j}^{(1)}(y). \quad (2.11)$$

From this representation we immediately obtain the following lemma:

**Lemma 1.** (see ref. [19]) *If for some integer  $j$ , the series  $\Sigma_{\vec{a};\vec{b};j}^{(1)}(u)$  is expressible in terms of Remiddi-Vermaseren functions (A.4b) with rational coefficients, then the sums  $\Sigma_{\vec{a};\vec{b};j+i}^{(1)}(u)$  for positive integers  $i$  can also be expressed in terms of functions of this type with rational coefficients.*

In a similar manner, let us rewrite the differential equations for the generating function of the *multiple binomial sums* as

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^c\Sigma_{\vec{a};\vec{b};c}^{(-1)}(\chi)=\frac{1+\chi}{1-\chi}\sigma_{\vec{a};\vec{b}}^{(-1)}(\chi), \quad (2.12a)$$

$$\frac{1}{2}(1+\chi)^2\frac{d}{d\chi}\sigma_{\vec{a};\vec{b}}^{(-1)}(\chi)=\delta_{p,0}+\left(2\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}+1\right)R_{\vec{a};\vec{b}}^{(-1)}(\chi). \quad (2.12b)$$

The first equation may be rewritten as

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^{c-j}\Sigma_{\vec{a};\vec{b};c}^{(-1)}(\chi)=\Sigma_{\vec{a};\vec{b};j}^{(-1)}(\chi), \quad (2.13)$$

or in an equivalent form

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^{c-j-1}\Sigma_{\vec{a};\vec{b};c}^{(-1)}(\chi)=\int_0^\chi d\chi\left(\frac{1}{\chi}-\frac{2}{1+\chi}\right)\Sigma_{\vec{a};\vec{b};j}^{(-1)}(\chi). \quad (2.14)$$

In this case, the boundary condition (2.5) is unchanged, and we can make a statement similar to the previous one:

**Lemma 2.** (see ref. [19]) *If for some integer  $j$ , the series  $\Sigma_{\vec{a};\vec{b};j}^{(-1)}(u)$  is expressible in terms of harmonic polylogarithms (A.4b) with rational coefficients, then the sums  $\Sigma_{\vec{a};\vec{b};j+i}^{(-1)}(u)$  for positive integers  $i$  can also be expressed in terms of harmonic polylogarithms with rational coefficients.*

## 2.2 Analytical evaluation of multiple (inverse) binomial sums of arbitrary weight and depth

Let us now consider the special case of sums (2.1) including only products of harmonic sums with argument  $(j-1)$ ; see eq. (1.1), and show that they are expressible in terms

of Remiddi-Vermaseren functions.<sup>7</sup> In agreement with ref. [12], we will denote such a sum as  $\Sigma_{a_1, \dots, a_p; -; m}^{(k)}(u)$ . In this case, the non-homogeneous term  $r_{\vec{a}; -}^{(k)}(j)$  of differential equation (2.4a) is again expressible in terms of sums of the same type,  $\Sigma_{b_1, \dots, b_p; -; m}^{(k)}(u)$ , but with smaller *depth*:

$$\binom{2j}{j}^k r_{\vec{a}; -}^{(k)}(j) = \prod_{r=1}^p [S_{a_r}(j-1) + j^{-a_r}] - \prod_{r=1}^p S_{a_r}(j-1), \quad k = \pm 1. \quad (2.15)$$

We shall start with the case of *inverse binomial sums*,  $k = 1$ :

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1).$$

In order to prove theorem A for inverse binomial sums, we will prove an auxiliary proposition:

**Proposition 1.** *For  $c = 1$ , the inverse binomial sums are expressible in terms of harmonic polylogarithms with rational coefficients  $c_{r, \vec{s}}$  times a factor  $(1-y)/(1+y)$ :*

$$\Sigma_{a_1, \dots, a_p; -; 1}^{(1)}(u) \Big|_{u=u(y)} = \frac{1-y}{1+y} \sum_{r, \vec{s}} c_{r, \vec{s}} \ln^r y \operatorname{Li}_{(\vec{s})}(y), \quad (2.16)$$

where  $r + s_1 + \dots + s_k = 1 + a_1 + \dots + a_p$  (weight of l.h.s. = weight of r.h.s.).

Substituting expression (2.16) in the r.h.s. of eq. (2.11), setting  $j = 1$ , and making trivial splitting of the denominator, we get the following result:

**Corollary 1.** *For  $c \geq 2$ , the inverse binomial sums are expressible in terms of harmonic polylogarithms with rational coefficients  $d_{r, \vec{s}}$ :*

$$\Sigma_{a_1, \dots, a_p; -; c}^{(1)}(u) \Big|_{u=u(y)} = \sum_{r, \vec{s}} d_{r, \vec{s}} \ln^r y \operatorname{Li}_{(\vec{s})}(y), \quad c \geq 2 \quad (2.17)$$

where  $r + s_1 + \dots + s_k = c + a_1 + \dots + a_p$  (weight of l.h.s. = weight of r.h.s.).

*Proof.* Let us consider *inverse binomial sums* of *depth 0*:

$$\Sigma_{-; -; c}^{(1)}(u) \equiv \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c}.$$

It was shown in ref. [8] that for any  $c \geq 2$  this sum is expressible in terms of generalized log-sine functions [37] which could be rewritten [10, 42] in terms of Nielsen polylogarithms. [38]

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<sup>7</sup>These sums are related to the multiple sums

$$\sum_{n_1 > n_2 > \dots > n_p = 1}^{\infty} \frac{1}{\binom{2n_1}{n_1}} \frac{u^{n_1}}{n_1^{c_1} n_2^{c_2} \dots n_p^{c_p}}.$$



Here we will present an iterated solution for the case of interest. The system (2.9) has the form

$$\left(\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-1}\Sigma_{-;-;c}^{(1)}(y)=\frac{1-y}{1+y}\sigma_{-;-}^{(1)}(y), \tag{2.18a}$$

$$y\frac{d}{dy}\sigma_{-;-}^{(1)}(y)=1. \tag{2.18b}$$

For  $c = 1$ , we immediately get the relation

$$\Sigma_{-;-;1}^{(1)}(y)=\frac{1-y}{1+y}\ln y, \tag{2.19}$$

which coincides with proposition 1 and can be readily transformed into the form of eq. (2.10):

$$\left(\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-2}\Sigma_{-;-;c}^{(1)}(y)=-\frac{1}{2}\ln^2 y. \tag{2.20}$$

The iterated solution of this differential equation for an arbitrary integer  $c \geq 2$  is expressible in terms of Remiddi-Vermaseren functions with rational coefficients<sup>8</sup> (in accordance with corollary 1).

For sums of *depth 1*, i.e.

$$\Sigma_{a_1;-;c}^{(1)}(u)\equiv\sum_{j=1}^{\infty}\frac{1}{\binom{2j}{j}}\frac{u^j}{j^c}S_{a_1}(j-1)\equiv\sum_{j=1}^{\infty}\frac{1}{\binom{2j}{j}}\frac{u^j}{j^c}\sum_{i=1}^{j-1}\frac{1}{i^{a_1}},$$

the coefficients of the non-homogeneous part are equal to *inverse binomial sums* of the zero depth,  $\binom{2j}{j}r_{a_1;-}^{(1)}(j)=1/j^{a_1}$ , and eqs. (2.9) take the form

$$\left(\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-1}\Sigma_{a_1;-;c}^{(1)}(y)=\frac{1-y}{1+y}\sigma_{a_1;-}^{(1)}(y), \tag{2.21a}$$

$$y\frac{d}{dy}\sigma_{a_1;-}^{(1)}(y)=\Sigma_{-;-;a_1}^{(1)}(y). \tag{2.21b}$$

For  $c = 1$  the system of equations (2.21) takes the simplest form

$$\Sigma_{a_1;-;1}^{(1)}(y)=\frac{1-y}{1+y}\sigma_{a_1;-}^{(1)}(y), \tag{2.22a}$$

$$y\frac{d}{dy}\sigma_{a_1;-}^{(1)}(y)=\Sigma_{-;-;a_1}^{(1)}(y). \tag{2.22b}$$

Let us now consider the case  $a_1 = 1$ . Using eq. (2.19), we derive from eq. (2.22)

$$\sigma_{1;-}^{(1)}(y)=\frac{1}{2}\ln^2 y-2\ln y\ln(1+y)-2\text{Li}_2(-y),$$

i.e., result is expressible in terms of harmonic polylogarithms. For  $a_1 \geq 2$ , the r.h.s. of the second equation (2.22b) is expressible in terms of harmonic polylogarithms with

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<sup>8</sup>Compare with the results of refs. [8, 10, 42].

rational coefficients (in accordance with previous considerations), so that  $\sigma_{a_1;-}^{(1)}(y)$  will also be expressible in terms of harmonic polylogarithms with rational coefficients. Substituting these results in the first equation (2.22a), we obtain results in accordance with proposition 1. For  $c \geq 2$ , the desired result follows from lemma 1.

We may complete the proof by mathematical induction. Let us assume that proposition 1 is valid for *multiple inverse binomial sums of depth k*:

$$\begin{aligned} \Sigma_{a_1, \dots, a_k; -; 1}^{(1)}(u) &\equiv \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=u(y)} \\ &= \frac{1-y}{1+y} \sum_{r, \vec{s}} c_{r, \vec{s}} \ln^r y \operatorname{Li}_{(\vec{s})}(y) , \end{aligned} \tag{2.23}$$

where  $\operatorname{Li}_{(\vec{s})}(z)$  is a coloured polylogarithm of a square root of unity,  $\vec{s} = s_1, \dots, s_k$ , and  $r + s_1 + \dots + s_p = c + a_1 + \dots + a_k$ . Then for  $c \geq 2$ , corollary 1 also holds for *multiple inverse binomial sums of depth k*:

$$\Sigma_{a_1, \dots, a_k; -; c}^{(1)}(u) \equiv \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=u(y)} = \sum_{r, \vec{s}} \tilde{c}_{r, \vec{s}} \ln^r y \operatorname{Li}_{(\vec{s})}(y) , \tag{2.24}$$

For the sum of *depth k+1*, the coefficients of the non-homogeneous part may be expressed as linear combinations of sums of *depth j*,  $j = 0, \dots, k$ , with integer coefficients and all possible symmetric distributions of the original indices between terms of the new sums:

$$\left( -\frac{1-y}{1+y} y \frac{d}{dy} \right)^{c-1} \Sigma_{a_1, \dots, a_{k+1}; -; c}^{(1)}(y) = \frac{1-y}{1+y} \sigma_{a_1, \dots, a_{k+1}; -}^{(1)}(y) , \tag{2.25a}$$

$$y \frac{d}{dy} \sigma_{a_1, \dots, a_{k+1}; -}^{(1)}(y) = \sum_{j=1}^{\infty} \frac{u^j}{\binom{2j}{j}} \sum_{p=0}^k \sum_{(i_1, \dots, i_{k+1})} \frac{1}{p!(k+1-p)!} \frac{S_{i_1}(j-1) \cdots S_{i_p}(j-1)}{j^{i_{p+1} + \dots + i_{k+1}}} , \tag{2.25b}$$

where the sum over indices  $(i_1, \dots, i_{k+1})$  is to be taken over all permutations of the list  $(a_1, \dots, a_{k+1})$ . If  $i_{p+1} + \dots + i_{k+1} \geq 2$ , the r.h.s. of eq. (2.25b) is expressible in terms of harmonic polylogarithms of *weight k* with rational coefficients; see eq. (2.24). As the result of integrating this equation,  $\sigma_{a_1, \dots, a_{k+1}; -}^{(1)}(y)$  also will be expressible in terms of harmonic polylogarithms of *weight k+1* with rational coefficients.

If  $i_{p+1} + \dots + i_{k+1} = 1$ , the r.h.s. of eq. (2.25b) is expressible in terms of harmonic polylogarithms of *weight k* with a common factor  $(1-y)/(1+y)$ ; see eq. (2.23). The result of integrating this equation again will be expressible in terms of harmonic polylogarithms of *weight k+1* with rational coefficients:

$$\sigma_{a_1, \dots, a_{k+1}; -}^{(1)}(y) = \int_1^y dt \left( \frac{1}{t} - \frac{2}{1+t} \right) \sum_{r, \vec{s}} c_{r, \vec{s}} \ln^r t \operatorname{Li}_{(\vec{s})}(t) .$$

For  $c = 1$ , direct substitution of the previous results into (2.25a) will show that proposition 1 is valid at *weight*  $k+1$ . In this way, the proposition 1 is proven for all weights. Then for  $c \geq 2$ , corollary 1 is also true for *multiple inverse binomial sums* of *depth*  $k+1$ .

Applying the differential operator  $u \frac{d}{du} \equiv -\frac{1-y}{1+y} y \frac{d}{dy}$  repeatedly  $l$  times to the sum  $\Sigma_{a_1, \dots, a_p; -; c}^{(1)}(u)$ , we can derive results for a similar sum with  $c \leq 1$ .<sup>9</sup> Thus, theorem A is proven for *multiple inverse binomial sums*.<sup>10</sup>

Let us now consider the *multiple binomial sums*,<sup>11</sup> ( $k = -1$ ),  $\Sigma_{a_1, \dots, a_p; -; c}^{(-1)}(u)$

$$\sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1).$$

In order to prove theorem A for binomial sums, we will first prove the following auxiliary proposition:

**Proposition 2.** *For  $c = 0$ , the binomial sums are expressible in terms of harmonic polylogarithms and have the following structure:*

$$\Sigma_{a_1, \dots, a_p; -; 0}^{(-1)}(u) \Big|_{u=u(\chi)} = \sum_{r, \vec{s}} \left[ \frac{1}{1-\chi} c_{r, \vec{s}} + d_{r, \vec{s}} \right] \ln^r \chi \operatorname{Li}_{(\vec{s})}(\chi), \quad (2.26)$$

where  $r + s_1 + \cdots + s_k = 1 + a_1 + \cdots + a_p$  (*weight of l.h.s. = weight of r.h.s.*) and  $c_{r, \vec{s}}$  and  $d_{r, \vec{s}}$  are rational numbers.

Substituting the expression (2.26) in the r.h.s. of eq. (2.14) and setting  $j = 0$ , we get

**Corollary 2.** *For  $c \geq 1$ , the binomial sums are expressible in terms of harmonic polylogarithms with rational coefficients  $\tilde{d}_{r, \vec{s}}$ :*

$$\Sigma_{a_1, \dots, a_p; -; c}^{(-1)}(u) \Big|_{u=u(\chi)} = \sum_{r, \vec{s}} \tilde{d}_{r, \vec{s}} \ln^r \chi \operatorname{Li}_{(\vec{s})}(\chi), \quad c \geq 1, \quad (2.27)$$

where  $r + s_1 + \cdots + s_k = c + a_1 + \cdots + a_p$  (*weight of l.h.s. is equal to weight of r.h.s.*).

We start again from the *multiple binomial sums* of *depth* 0,

$$\Sigma_{-; -; c}^{(-1)}(u) \equiv \sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c}.$$

<sup>9</sup>Some particular cases of sums of this type were considered also in ref. [43].

<sup>10</sup>All multiple inverse binomial sums up to *weight* 4 were calculated in ref. [12]; see table I in appendix C.

<sup>11</sup>These sums are related to the multiple sums

$$\sum_{n_1 > n_2 > \cdots > n_p = 1}^{\infty} \binom{2n_1}{n_1} \frac{u^{n_1}}{n_1^c n_2^{b_1} \cdots n_p^{b_p}}.$$

In this case, eqs. (2.12) have the form

$$\begin{aligned} \left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^c \Sigma_{-;-;c}^{(-1)}(\chi) &= \frac{1+\chi}{1-\chi}\sigma_{-;-}^{(-1)}(\chi), \\ \frac{1}{2}(1+\chi)^2\frac{d}{d\chi}\sigma_{-;-}^{(-1)}(\chi) &= 1, \end{aligned} \tag{2.28a}$$

where the factor  $\frac{1+\chi}{1-\chi}$  may be written as

$$\frac{1+\chi}{1-\chi} = \frac{2}{1-\chi} - 1. \tag{2.29}$$

For  $c = 0$ , we obtain

$$\Sigma_{-;-;0}^{(-1)}(\chi) = 2 \left[ \frac{1}{1-\chi} - 1 \right], \tag{2.30}$$

which coincides with proposition 2. Substituting this result into r.h.s. of eq. (2.14) we find

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^{c-1} \Sigma_{-;-;c}^{(-1)}(\chi) = 2 \ln(1+\chi). \tag{2.31}$$

The results of iterated integration, for  $c \geq 1$  and boundary condition defined by eq. (2.5), are expressible in terms of *generalized polylogarithms* (A.2) with rational coefficients (see Corollary B). For the sums of *depth 1*,

$$\Sigma_{a_1;-;c}^{(-1)}(u) \equiv \sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} S_{a_1}(j-1) \equiv \sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} \sum_{i=1}^{j-1} \frac{1}{i^{a_1}},$$

we have

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^c \Sigma_{a_1;-;c}^{(-1)}(\chi) = \frac{1+\chi}{1-\chi}\sigma_{a_1;-}^{(-1)}(\chi), \tag{2.32a}$$

$$\frac{1}{2}(1+\chi)^2\frac{d}{d\chi}\sigma_{a_1;-}^{(-1)}(\chi) = 2\Sigma_{-;-;a_1-1}^{(-1)}(\chi) + \Sigma_{-;-;a_1}^{(-1)}(\chi). \tag{2.32b}$$

Integrating by part in eq. (2.32b) and taking into account that<sup>12</sup>

$$\frac{d}{d\chi}\Sigma_{-;-;c}^{(-1)}(\chi) = \left(\frac{1}{\chi} - \frac{2}{1+\chi}\right)\Sigma_{-;-;c-1}^{(-1)}(\chi),$$

we obtain

$$\sigma_{a_1;-}^{(-1)}(\chi) = -\frac{1-\chi}{1+\chi}\Sigma_{-;-;a_1}^{(-1)}(\chi) + \int_0^\chi \frac{dt}{t}\Sigma_{-;-;a_1-1}^{(-1)}(t). \tag{2.33}$$

Using this results in the r.h.s. of eq. (2.32a), we have

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^c \Sigma_{a_1;-;c}^{(-1)}(\chi) = -\Sigma_{-;-;a_1}^{(-1)}(\chi) + \frac{1+\chi}{1-\chi} \int_0^\chi \frac{dt}{t}\Sigma_{-;-;a_1-1}^{(-1)}(t). \tag{2.34}$$

<sup>12</sup>This relation follows from the differential relation

$$u\frac{d}{du}\Sigma_{\bar{a};\bar{b};c}^{(k)}(u) = \Sigma_{\bar{a};\bar{b};c-1}^{(k)}(u).$$

Let us set  $c = 0$ . It is necessary to consider two cases: (i)  $a_1 = 1$  and (ii)  $a_1 \geq 2$ . For  $a_1 = 1$ , we can use the explicit results (2.30) and (2.31) to get

$$\Sigma_{1;-;0}^{(-1)}(\chi) = 2 \ln(1 - \chi) - 2 \ln(1 + \chi) - \frac{4}{1 - \chi} \ln(1 - \chi) ,$$

in accordance with proposition 2. For  $a_1 \geq 2$  the r.h.s. of eq. (2.32b) is expressible in terms of harmonic polylogarithms with rational coefficients, so that eq. (2.34) is also expressible in terms of harmonic polylogarithms with rational coefficients in accordance with proposition 2.

For  $c \geq 1$ , the desired result follows from lemma 2:

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^{c-1}\Sigma_{a_1;-;c}^{(-1)}(\chi) = -\Sigma_{-;-;a_1+1}^{(-1)}(\chi) + \int_0^\chi \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \Sigma_{-;-;a_1-1}^{(-1)}(t_2) . \quad (2.35)$$

In particular, for  $a_1 = 1$  we have

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^{c-1}\Sigma_{1;-;c}^{(-1)}(\chi) = 2\text{Li}_2(-\chi) + 2\ln^2(1 + \chi) + 2\text{Li}_2(\chi) . \quad (2.36)$$

Let us assume proposition 2 is valid for *multiple binomial sums* of *depth*  $k$ , and prove the proposition for *depth*  $k+1$ . Thus, we assume that

$$\begin{aligned} \Sigma_{a_1,\dots,a_k;-;0}^{(-1)}(u) &\equiv \sum_{j=1}^{\infty} \binom{2j}{j} u^j S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Big|_{u=\chi} \\ &= \sum_{p,\vec{s}} \left[ \frac{1}{1-\chi} c_{p,\vec{s}} + d_{p,\vec{s}} \right] \ln^p \chi \text{Li}_{\left(\frac{\vec{s}}{\vec{s}}\right)}(\chi) , \end{aligned} \quad (2.37)$$

where  $\text{Li}_{\left(\frac{\vec{s}}{\vec{s}}\right)}(\chi)$  is a coloured polylogarithm of a square root of unity,  $\vec{s} = (s_1, \dots, s_k)$ , and  $p + s_1 + \dots + s_p = a_1 + \dots + a_k$ . Then for  $c \geq 1$ , corollary 2 also holds for *multiple binomial sums* of *depth*  $k$ :

$$\begin{aligned} \Sigma_{a_1,\dots,a_k;-;c}^{(-1)}(u) &\equiv \sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Big|_{u=\chi} \\ &= \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p \chi \text{Li}_{\left(\frac{\vec{s}}{\vec{s}}\right)}(\chi) , \end{aligned} \quad (2.38)$$

For a sum of *depth*  $k+1$ , the coefficients of the non-homogeneous part are expressed as linear combinations of sums of *depth*  $j$ ,  $j = 0, \dots, k$ , with an integer coefficients and all possible distributions of the original indices between terms of new sums, multiplied by a factor  $(2j + 1)$ :

$$\left(\frac{1+\chi}{1-\chi}\chi\frac{d}{d\chi}\right)^c \Sigma_{a_1,\dots,a_{k+1};-;c}^{(-1)}(\chi) = \frac{1+\chi}{1-\chi} \sigma_{a_1,\dots,a_{k+1};-}^{(-1)}(\chi) , \quad (2.39a)$$

$$\begin{aligned} \frac{1}{2}(1+\chi)^2 \frac{d}{d\chi} \sigma_{a_1,\dots,a_{k+1};-}^{(-1)}(\chi) &= \sum_{j=1}^{\infty} (2j+1) \binom{2j}{j} u^j \\ &\times \sum_{p=0}^k \sum_{(i_1,\dots,i_{k+1})} \frac{1}{p!(k+1-p)!} \frac{S_{i_1}(j-1) \cdots S_{i_p}(j-1)}{j^{i_{p+1}+\dots+i_{k+1}}} , \end{aligned} \quad (2.39b)$$

where the sum over indices  $(i_1, \dots, i_{k+1})$  is to be taken over all permutations of the list  $(a_1, \dots, a_{k+1})$ .

Let us denote the sub-list of length  $p$  as  $\vec{I} = (i_1, \dots, i_p)$  and define the sum of the remaining indices as  $J = i_{p+1} + \dots + i_{k+1}$ , so that the second equation (2.39b) can be written as

$$\frac{1}{2}(1+\chi)^2 \frac{d}{d\chi} \sigma_{a_1, \dots, a_{k+1}; -}^{(-1)}(\chi) = \sum_{\vec{I}, J} \left[ 2\Sigma_{\vec{I}; -; J}^{(-1)}(\chi) + \Sigma_{\vec{I}; -; J-1}^{(-1)}(\chi) \right].$$

Integrating by parts, we find

$$\sigma_{a_1, \dots, a_{k+1}; -}^{(-1)}(\chi) = \sum_{\vec{I}, J} \left[ -\frac{1-\chi}{1+\chi} \Sigma_{\vec{I}; -; J}^{(-1)}(\chi) + \int_0^\chi \frac{dt}{t} \Sigma_{\vec{I}; -; J-1}^{(-1)}(t) \right].$$

Substituting this result into the r.h.s. of eq. (2.39a) we have

$$\left( \frac{1+\chi}{1-\chi} \chi \frac{d}{d\chi} \right)^c \Sigma_{a_1, \dots, a_{k+1}; -; c}^{(-1)}(\chi) = \sum_{\vec{I}, J} \left[ -\Sigma_{\vec{I}; -; J}^{(-1)}(\chi) + \frac{1+\chi}{1-\chi} \int_0^\chi \frac{dt_1}{t_1} \Sigma_{\vec{I}; -; J-1}^{(-1)}(t_1) \right]. \quad (2.40)$$

Let us set  $c = 0$  and consider two cases: (i)  $J = 1$  and (ii)  $J \geq 2$ . For  $J = 1$ , the first term of the r.h.s. of eq. (2.40) is expressible in terms of harmonic polylogarithms with rational coefficients. The last term of the r.h.s. of eq. (2.40) has the structure of eq. (2.37) so that after integration, it will again be expressible in terms of harmonic polylogarithms of *weight*  $k+1$ . For  $J \geq 2$ , both terms of the r.h.s. of eq. (2.40) are expressible in terms of harmonic polylogarithms of *weight*  $k+1$  (see eq. (2.40)). In this way, the proposition 2 is found to be valid at the *weight*  $k+1$ . Consequently, proposition 2 is proven for all weights. Therefore, for  $c \geq 1$ , corollary 2 is also valid for the *multiple binomial sums* of *weight*  $k+1$ .

Applying the differential operator  $u \frac{d}{du} = \frac{1+\chi}{1-\chi} \chi \frac{d}{d\chi}$  repeatedly  $l$  times to the sum  $\Sigma_{a_1, \dots, a_p; -; c}^{(-1)}(\chi)$ , we can derive results for similar sums with  $c \leq 0$ . Thus, theorem A is proven for *multiple binomial sums*.<sup>13</sup>

### 3. All-order $\epsilon$ -expansion of hypergeometric functions with one half-integer value of the parameters via multiple (inverse) binomial sums

In this section, we turn our attention to the proof of theorem B. It is well known that any function  ${}_pF_{p-1}(\vec{a} + \vec{m}; \vec{b} + \vec{k}; z)$  is expressible in terms of  $p$  other functions of the same type:

$$R_{p+1}(\vec{a}, \vec{b}, z) {}_pF_{p-1}(\vec{a} + \vec{m}; \vec{b} + \vec{k}; z) = \sum_{k=1}^p R_k(\vec{a}, \vec{b}, z) {}_pF_{p-1}(\vec{a} + \vec{e}_k; \vec{b} + \vec{E}_k; z), \quad (3.1)$$

where  $\vec{m}, \vec{k}, \vec{e}_k$ , and  $\vec{E}_k$  are lists of integers and  $R_k$  are polynomials in parameters  $\vec{a}, \vec{b}$ , and  $z$ . Systematic methods for solving this problem were elaborated in refs. [44, 45]. For generalized hypergeometric functions of theorem B, let us choose as basis functions arbitrary  $p$ -functions from the following set:

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<sup>13</sup>All multiple binomial sums up to *weight* 3 were calculated in ref. [11, 12]; see the proper appendixes.

- for eq. (1.2a) there are  $p^2$  functions of the proper type:

$${}_pF_{p-1} \left( \begin{matrix} \frac{3}{2}, \{1 + a_i \varepsilon\}^{p-L-1}, \{2 + d_i \varepsilon\}^L \\ \{1 + e_i \varepsilon\}^{p-Q-1}, \{2 + c_i \varepsilon\}^Q \end{matrix} \middle| z \right),$$

- for eq. (1.2b) there are  $p^2 - 1$  functions of the proper type:

$${}_pF_{p-1} \left( \begin{matrix} \{1 + a_i \varepsilon\}^{p-L}, \{2 + d_i \varepsilon\}^L \\ \frac{3}{2}, \{1 + e_i \varepsilon\}^{p-Q-2}, \{2 + c_i \varepsilon\}^Q \end{matrix} \middle| z \right).$$

In the framework of the approach developed in refs. [8, 10–12, 19], the study of the  $\varepsilon$ -expansion of basis hypergeometric functions has been reduced to the study of multiple (*inverse*) binomial sums. It is easy to get the following representations:

$${}_pF_{p-1} \left( \begin{matrix} \{1 + a_i \varepsilon\}^K, \{2 + d_i \varepsilon\}^L \\ \frac{3}{2}, \{1 + e_i \varepsilon\}^R, \{2 + c_i \varepsilon\}^Q \end{matrix} \middle| z \right) = \frac{1}{2z} \frac{\prod_{s=1}^Q (1 + c_s \varepsilon)}{\prod_{i=1}^L (1 + d_i \varepsilon)} \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{(4z)^j}{j^{K-R-1}} \Delta, \quad (3.2a)$$

$${}_pF_{p-1} \left( \begin{matrix} \frac{3}{2}, \{1 + a_i \varepsilon\}^K, \{2 + d_i \varepsilon\}^L \\ \{1 + e_i \varepsilon\}^R, \{2 + c_i \varepsilon\}^Q \end{matrix} \middle| z \right) = \frac{2}{z} \frac{\prod_{s=1}^Q (1 + c_s \varepsilon)}{\prod_{i=1}^L (1 + d_i \varepsilon)} \sum_{j=1}^{\infty} \binom{2j}{j} \frac{\left(\frac{z}{4}\right)^j}{j^{K-R-1}} \Delta, \quad (3.2b)$$

where the superscripts  $K, L, R, Q$  show the lengths of the parameter lists,

$$\Delta = \exp \left[ \sum_{k=1}^{\infty} \frac{(-\varepsilon)^k}{k} \left( w_k j^{-k} + S_k(n-1)t_k \right) \right] = 1 - \varepsilon \left( \frac{w_1}{j} + t_1 S_1(n-1) \right) + \mathcal{O}(\varepsilon^2), \quad (3.3)$$

$S_a(n) = \sum_{j=1}^n 1/j^a$  is a harmonic sum, and the constants are defined as

$$\begin{aligned} A_k &\equiv \sum a_i^k, & C_k &\equiv \sum c_i^k, & D_k &\equiv \sum d_i^k, & E_k &\equiv \sum e_i^k, \\ t_k &\equiv C_k + E_k - A_k - D_k, & w_k &\equiv C_k - D_k, \end{aligned}$$

where the summations extend over all possible values of the parameters in eqs. (3.2). In this way, the  $\varepsilon$ -expansions of the basis functions (3.2) are expressible in terms of *multiple (inverse) binomial sums* studied in section 2. But all these are expressible in terms of harmonic polylogarithms. Thus, theorem B is proven.

#### 4. Generalized multiple (inverse) binomial sums via derivatives of generalized hypergeometric functions

In physical applications, in particular, within Smirnov-Tausk approach, more general sums, in addition to the ones defined in eq. (1.1), may be generated:

$$\sum_{j=1}^{\infty} \left[ \frac{(j + c_1)!(j + c_2)!}{(2j + c_3)!} \right]^k \frac{w^j}{(nj + c_4)^c} S_{a_1}(m_1 j + b_1) \cdots S_{a_k}(m_k j + b_k),$$

where  $\{a_i\}, \{b_j\}, \{c_k\}, \{m_k\}, n$  are integers and  $k = \pm 1$ . The procedure of finding the proper differential equation (see refs. [12, 36] for a detailed discussion) can be applied

to analytically evaluate any of these new sums. Another approach is based on extension of the algorithm of nested sums [17, 18] for the study of the algebraic relations between these sums. However, there is a third approach arising from the possibility of reducing an arbitrary generalized hypergeometric function to a set of basis functions with the help of the Zeilberger-Takayama algorithm described by eq. (3.1).

To be more specific, let us divide both sides of eq. (3.1) by  $R_{p+1}(a_i, b_j, z)$  and construct the  $\varepsilon$ -expansion for the hypergeometric functions described in theorem B. The r.h.s. of this relation is expressible analytically in terms of harmonic polylogarithms with polynomial coefficients. The l.h.s. can be used as a generating function for generalized multiple (inverse) binomial sums. Using a standard form for the Taylor expansion of the Gamma function,<sup>14</sup>

$$\frac{(m + a\varepsilon)_j}{(m)_j} = \exp \left\{ - \sum_{k=1}^{\infty} \frac{(-a\varepsilon)^k}{k} [S_k(m+j-1) - S_k(m-1)] \right\},$$

where  $(\alpha)_j \equiv \Gamma(\alpha + j)/\Gamma(\alpha)$  is the Pochhammer symbol, we obtain

$$\begin{aligned} {}^{P+1}F_P \left( \begin{matrix} \{m_l + a_l\varepsilon\}^L, \{p_i + \frac{1}{2}\}^{P+1-L} \\ \{n_k + b_k\varepsilon\}^K, \{q_j + \frac{1}{2}\}^{P-K} \end{matrix} \middle| z \right) &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\prod_{l=1}^L (m_l + a_l\varepsilon)_j}{\prod_{k=1}^K (n_k + b_k\varepsilon)_j} \frac{\prod_{i=1}^{P+1-L} (p_i + \frac{1}{2})_j}{\prod_{s=1}^{P-K} (q_s + \frac{1}{2})_j} \\ &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{1}{4^{j(K-L+1)}} \frac{\prod_{l=1}^L (m_l)_j}{\prod_{k=1}^K (n_k)_j} \prod_{i=1}^{P+1-L} \frac{(2p_i+1)_{2j}}{(p_i+1)_j} \prod_{s=1}^{P-K} \frac{(l_s+1)_j}{(2l_s+1)_{2j}} \Delta, \end{aligned} \quad (4.1)$$

where the  $m_l, n_k, p_i, q_j$  are integers and

$$\begin{aligned} \Delta = \exp \left[ \sum_{k=1}^{\infty} \frac{(-\varepsilon)^k}{k} \left( \sum_{\omega=1}^K b_{\omega}^k [S_k(n_{\omega} + j - 1) - S_k(n_{\omega} - 1)] \right. \right. \\ \left. \left. - \sum_{i=1}^L [a_i^k S_k(m_i + j - 1) - a_i^k S_k(m_i - 1)] \right) \right]. \end{aligned}$$

Setting  $K = L = P$  in eq. (4.1), we get generating functions for generalized multiple binomial sums: the derivatives

$$\prod_{l,k} \left( \frac{\partial}{\partial a_l} \right)^{r_l} \left( \frac{\partial}{\partial b_k} \right)^{s_k} {}^{P+1}F_P \left( \begin{matrix} \{a_l\}^P, p + \frac{1}{2} \\ \{b_k\}^P \end{matrix} \middle| z \right) \Bigg|_{a_l=m_l; b_k=n_k} \quad (4.2a)$$

lead to terms in the epsilon expansion of the form

$$\sum_{j=0}^{\infty} \frac{(2p+1)_{2j}}{(p+1)_j} \frac{1}{j!} \frac{z^j}{4^j} \frac{\prod_{l=1}^P (m_l)_j}{\prod_{k=1}^P (n_k)_j} \prod_{M=1} S_{a_M}(I_M + j), \quad (4.2b)$$

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<sup>14</sup>The relation between harmonic sums  $S_a(j)$  and derivatives of the function  $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$  is

$$\psi^{(k-1)}(j) = (-1)^k (k-1)! [\zeta_k - S_k(j-1)], \quad k > 1.$$



where the  $I_M$  are integers from the lists  $\{m_l\}^L$  and  $\{n_k\}^K$ . For  $L = P + 1$  and  $K = P - 1$  we get generating functions for generalized multiple inverse binomial sums:

$$\prod_{l,k} \left(\frac{\partial}{\partial a_l}\right)^{r_l} \left(\frac{\partial}{\partial b_k}\right)^{s_k} {}_{P+1}F_P \left( \begin{matrix} \{a_l\}^{P+1} \\ \{b_k\}^{P-1}, q + \frac{1}{2} \end{matrix} \middle| z \right) \Bigg|_{a_l=m_l; b_k=n_k} \Rightarrow \sum_{j=0}^{\infty} \frac{(q+1)_j}{(2q+1)_{2j}} \frac{(4z)^j}{j!} \frac{\prod_{l=1}^{P+1} (m_l)_j}{\prod_{k=1}^{P-1} (n_k)_j} \prod_{M=1} S_{a_M}(I_M + j). \quad (4.3)$$

For  $K = P$  and  $L = P + 1$  we get generating functions for generalized multiple harmonic sums:

$$\prod_{l,k} \left(\frac{\partial}{\partial a_l}\right)^{r_l} \left(\frac{\partial}{\partial b_k}\right)^{s_k} {}_{P+1}F_P \left( \begin{matrix} \{a_l\}^{P+1} \\ \{b_k\}^P \end{matrix} \middle| z \right) \Bigg|_{a_l=m_l; b_k=n_k} \Rightarrow \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\prod_{l=1}^{P+1} (m_l)_j}{\prod_{k=1}^P (n_k)_j} \prod_{M=1} S_{a_M}(I_M + j). \quad (4.4)$$

Instead of one hypergeometric function, we could consider a linear combination of the functions of the same type. Such a combination is also reducible and expressible in terms of our basis functions. Combining the proper set of hypergeometric functions, we could expect that any individual sums,<sup>15</sup> of the type described by r.h.s. of eqs. (4.2)–(4.4) are expressible in terms of generalized (harmonic) polylogarithms with polynomial coefficients.<sup>16</sup>

<sup>15</sup>Using the results of the all-order  $\varepsilon$ -expansion for Gauss hypergeometric functions [13, 29] we could consider a series of type (2.1).

<sup>16</sup>In particular, all sums presented in ref. [46] are reducible in terms of our basis sums or sums studied in ref. [12]. Indeed, taking into account that

$$(2n + 1) \binom{2n}{n} = \frac{n + 1}{2} \binom{2n + 2}{n + 1}$$

and shifting the index of summation we have

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{z^n}{(2n+1)} X_{\vec{a}}(n) Y_{\vec{b}}(2n+1) = \frac{2}{z} \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{z^j}{j} X_{\vec{a}}(j-1) Y_{\vec{b}}(2j-1) - X_{\vec{a}}(0) Y_{\vec{b}}(1), \quad (4.5)$$

where  $X_{\vec{a}}(n) = \prod_{k=1}^r S_{a_k}(n)$  and  $Y_{\vec{a}}(2n+1) = \prod_{k=1}^r S_{a_k}(2n+1)$ , are products of harmonic sums, with the vector  $\vec{a}$  having  $r$  components. As a consequence,  $X_{\vec{a}}(0) = 0$  and  $Y_{\vec{b}}(1) = 1$ . In this way, any sums described by eq. (4.5) may be reduced to sums of type (2.1), and for  $Y_{\vec{b}}(j) = 1$  they are reduced to the sums studied in the present paper. Another possible generalization of the sums considered here is

$$\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n}} \frac{z^n}{(2n+1)} X_{\vec{a}}(n+1) Y_{\vec{b}}(2n+1) = \frac{2}{z} \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{z^j}{j} X_{\vec{a}}(j) Y_{\vec{b}}(2j-1) - X_{\vec{a}}(1) Y_{\vec{b}}(1). \quad (4.6)$$

Due to the depth reduction relation,

$$X_{\vec{a}}(j) = X_{\vec{a}}(j-1) + \sum_{p=0}^r \sum_{(i_1, \dots, i_{k+1})} \frac{1}{p!(r-p)!} \frac{S_{i_1}(j-1) \cdots S_{i_p}(j-1)}{j^{i_{p+1} + \dots + i_{k+1}}},$$

sums of type (4.6) are also expressible in terms sums of type (4.5).

These arguments suggest a criterion for what type of generalized multiple (inverse) binomial sum are expressible in terms of harmonic polylogarithms with coefficients that are ratios of polynomials. This is just the beginning of a general analysis, but the corresponding analysis for harmonic sums is already known to be valid. [17] Unfortunately, existing computer algebra algorithms [47] do not allow us to identify the multiple series with derivatives of hypergeometric functions or their combinations. It is still matter of personal experience, but this approach looks very promising and is worthy of further analysis.

### 5. Discussion and conclusions

We have constructed an iterative solution for *multiple (inverse) binomial sums* defined by eq. (1.1). It was shown that by the appropriate change of variables, defined by eqs. (2.6) and (2.7), the multiple (inverse) binomial sums are converted into harmonic polylogarithms (see theorem A). Symbolically, this may be expressed as

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{w^j}{j} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=u(y)} = \frac{1-y}{1+y} \sum_{p,\bar{s}} c_{p,\bar{s}} \ln^p y \operatorname{Li}_{(\frac{\sigma}{\bar{s}})}(y), \quad (5.1a)$$

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{w^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=u(y)} = \sum_{p,\bar{s}} \tilde{c}_{p,\bar{s}} \ln^p y \operatorname{Li}_{(\frac{\sigma}{\bar{s}})}(y), \quad c \geq 2 \quad (5.1b)$$

and

$$\sum_{j=1}^{\infty} \binom{2j}{j} w^j S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=u(\chi)} = \sum_{p,\bar{s}} \left[ \frac{c_{p,\bar{s}}}{1-\chi} + d_{p,\bar{s}} \right] \ln^p \chi \operatorname{Li}_{(\frac{\sigma}{\bar{s}})}(\chi), \quad (5.2a)$$

$$\sum_{j=1}^{\infty} \binom{2j}{j} \frac{w^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=u(\chi)} = \sum_{p,\bar{s}} \tilde{c}_{p,\bar{s}} \ln^p \chi \operatorname{Li}_{(\frac{\sigma}{\bar{s}})}(\chi), \quad c \geq 1 \quad (5.2b)$$

where  $c$  is a positive integer,  $c_{p,\bar{s}}$ ,  $\tilde{c}_{p,\bar{s}}$  and  $d_{p,\bar{s}}$  are rational coefficients, the *weight* of l.h.s. = *weight* of r.h.s.,  $\operatorname{Li}_{(\frac{\sigma}{\bar{s}})}(\chi)$  is the coloured multiple polylogarithm of a square root of unity,

$$S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a},$$

is a harmonic series. The mappings (5.1), (5.2) are defined in the radius of convergence of the l.h.s.:

$$|u| \leq \begin{cases} 4, & \text{inverse binomial} \\ \frac{1}{4}, & \text{binomial} \end{cases} \quad (5.3)$$

Unfortunately, one of the unsolved problem is the completeness of the representation (5.1), (5.2). In other words, is it possible to express all harmonic polylogarithms in terms of multiple (inverse) binomial sums? If not, what kind of sums must be added to

get a complete basis? Another problem beyond our present considerations is to find the algebraic relations among the sums.

From representation (5.1), (5.2), it is evident that some (or all, if the basis is complete) of the alternating or non-alternating<sup>17</sup> multiple Euler-Zagier sums (or multiple zeta values) [15], can be written in terms of multiple (inverse) binomial sums of special values of arguments. Two arguments where such a representation is possible are trivially obtained by setting the arguments of the harmonic polylogarithms  $y, \chi$  to  $\pm 1$ :

$$u = 4, \quad y = -1, \tag{5.5}$$

$$u = \frac{1}{4}, \quad \chi = 1. \tag{5.6}$$

Another such point<sup>18</sup>

$$u = -1, \quad y = \frac{3 - \sqrt{5}}{2} \tag{5.7}$$

has been discussed intensively in the context of Apéry-like expressions for Riemann zeta functions (see [48] and References therein). For two other points

$$u = 1, \quad y = \exp\left(i\frac{\pi}{3}\right), \tag{5.8}$$

$$u = 2, \quad y = i, \tag{5.9}$$

the relation between multiple inverse binomial sums and multiple zeta values was analysed mainly by the method of experimental mathematics. [49] Some of the relations are presented in ref. [50] and in the appendix of ref. [10].

Let us make a few comments about harmonic polylogarithms of a complex argument. For the case  $0 \leq u \leq 4$ , the variable  $y$  defined in (2.6) belongs to a complex unit circle,  $y = \exp(i\theta)$ . In this case, the coloured polylogarithms of a square root of unity can be split into real and imaginary parts as in the case of classical polylogarithms. [37] At the present, there is no commonly accepted notation for the new functions generated by such splitting. In ref. [50], the *multiple Glaishers* and *multiple Clausen* functions were introduced as the real and imaginary parts of generalized polylogarithms of complex unit argument. In ref. [10, 26, 27, 42], the splitting of Nielsen polylogarithms was analysed in detail. In this case, the real and imaginary parts are reduced to classical Clausen functions,  $\text{Cl}_j(\theta)$  and generalized log-sine functions  $\text{Ls}_j^{(k)}(\theta)$ . Ref. [51] attempts to classify new functions on the basis of new  $\text{LsLsc}_{i,j,k}(\theta)$ -functions.

In appendix A of ref. [12], the iterated representation for Remiddi-Vermaseren functions of complex unit was constructed. It was observed [8, 10, 12, 52] that the physically

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<sup>17</sup>Let us recall that multiple Euler-Zagier sums are defined as

$$\zeta(s_1, \dots, s_k; \sigma_1, \dots, \sigma_k) = \sum_{n_1 > n_2 > \dots > n_k > 0} \prod_{j=1}^k \frac{(\sigma_j)^{n_j}}{n_j^{s_j}}, \tag{5.4}$$

where  $\sigma_j = \pm 1$  and  $s_j > 0$ .  $\sigma = 1$  is called non-alternating and  $\sigma = -1$  is alternating sums, correspondingly.

<sup>18</sup>We are thankful to Andrei Davydychev for information about the relation between this point and the “golden ratio”, [37],  $\frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2$ .

interesting case, representing single-scale diagrams with two massive particle cuts, corresponds to Remiddi-Vermaseren functions (A.9) with argument equal to a primitive “sixth root of unity”,  $y = \exp(i\frac{\pi}{3})$ . This gives an explanation of the proper “basis of transcendental constants” constructed in refs. [52] and [10], and its difference from the proper basis of Broadhurst [16]. Of course, for numerical evaluation of harmonic polylogarithms of complex argument, only a series representation is necessary. [53]

Using the results of theorem A, we have proved theorems B about the all-order  $\varepsilon$ -expansion of a special class of hypergeometric functions. The proof includes two steps: (i) the algebraic reduction of generalized hypergeometric functions of the type specified in theorems B to basic functions and (ii) the algorithms for calculating the analytical coefficients of the  $\varepsilon$ -expansion of basic hypergeometric functions. The implementation of step (i) — the reduction algorithm — is based on general considerations performed in refs. [44, 45]. In step (ii), the algorithm is based on series representation of the basis hypergeometric functions defined by eq. (4.1). The coefficients of the  $\varepsilon$ -expansion are expressible in terms of multiple (inverse) binomial sums analyzed in theorem A.

Exploring the opportunity to reduce an arbitrary generalized hypergeometric function to a set of basis functions with the help of the Zeilberger-Takayama algorithm, we have presented in section 4 some arguments about one possible generalization of (inverse) binomial sums (see eq. (4.4)) which would be expressible in terms of harmonic polylogarithms with coefficients that are ratios of polynomials.

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## A. Zoo of special functions

For completeness, we will present the definition of a set of new functions, such as *multiple polylogarithms*<sup>19</sup>

$$\text{Li}_{k_1, k_2, \dots, k_n}(z_1, z_2, \dots, z_n) = \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}. \tag{A.1}$$

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<sup>19</sup>For a review, we recommended ref. [54].

Special cases of *multiple polylogarithms*<sup>20</sup> include *generalized polylogarithms*, defined by

$$\text{Li}_{k_1, k_2, \dots, k_n}(z) = \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}, \quad (\text{A.2})$$

and *coloured polylogarithms of a square root of unity*, defined by<sup>21</sup>

$$\text{Li}_{\left(\frac{\vec{\sigma}}{\vec{s}}\right)}(z) \equiv \text{Li}_{\left(\begin{smallmatrix} \sigma_1, \sigma_2, \dots, \sigma_k \\ s_1, s_2, \dots, s_n \end{smallmatrix}\right)}(z) = \sum_{m_1 > m_2 > \dots > m_n > 0} z^{m_1} \frac{\sigma_1^{m_1} \dots \sigma_n^{m_n}}{m_1^{s_1} m_2^{s_2} \dots m_n^{s_n}}, \quad (\text{A.3})$$

where  $\vec{s} = (s_1, \dots, s_n)$  and  $\vec{\sigma} = (\sigma_1, \dots, \sigma_n)$  are multi-indices and  $\sigma_k$  is a square root of unity,  $\sigma_k = \pm 1$ . The extension of *coloured polylogarithms of square root of unity* (A.3) by inclusion of powers of logarithms,  $\ln^k z$ , leads to *harmonic polylogarithms* or Remiddi-Vermaseren polylogarithms (or functions) [22]. These can be written in the following form:

$$H_{\vec{A}}(z) = \sum_{p, \vec{k}} c_{p, \vec{k}} \ln^p z \text{Li}_{k_1, k_2, \dots, k_n}(z), \quad (\text{A.4a})$$

$$H_{\vec{B}}(z) = \sum_{p, \vec{s}} c_{p, \vec{s}} \ln^p z \text{Li}_{\left(\frac{\vec{\sigma}}{\vec{s}}\right)}(z). \quad (\text{A.4b})$$

where in the first equation, (A.4a), the vector  $\vec{A}$  includes only components 0 and 1, and in the second, (A.4b),  $-1$  components are included. The coefficients  $c_{\vec{k}}$  and  $c_{\vec{s}}$  are rational numbers. In eq. (A.4) the *weight* of the l.h.s. = the *weight* of the r.h.s.

Recall that *generalized polylogarithms* (A.2) can be expressed as iterated integrals of the form

$$\text{Li}_{k_1, \dots, k_n}(z) = \int_0^z \underbrace{\frac{dt}{t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_1-1 \text{ times}} \circ \frac{dt}{1-t} \circ \dots \circ \underbrace{\frac{dt}{t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_n-1 \text{ times}} \circ \frac{dt}{1-t}, \quad (\text{A.5})$$

where, by definition

$$\int_0^z \underbrace{\frac{dt}{t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_1-1 \text{ times}} \circ \frac{dt}{1-t} = \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \dots \int_0^{t_{k_1-2}} \frac{dt_{k_1-1}}{t_{k_1-1}} \int_0^{t_{k_1-1}} \frac{dt_{k_1}}{1-t_{k_1}}. \quad (\text{A.6})$$

The integral (A.5) is an iterated Chen integral [55] w.r.t. the differential forms  $\omega_0 = dz/z$  and  $\omega_1 = \frac{dz}{1-z}$ , so that

$$\text{Li}_{k_1, \dots, k_n}(z) = \int_0^z \omega_0^{k_1-1} \omega_1 \dots \omega_0^{k_n-1} \omega_1. \quad (\text{A.7})$$

The coloured polylogarithms (eq. A.3) also have an iterated integral representation w.r.t. three differential forms,

$$\begin{aligned} \omega_0 &= \frac{dy}{y}, & \sigma &= 0, \\ \omega_\sigma &= \frac{\sigma dy}{1-\sigma y}, & \sigma &= \pm 1, \end{aligned} \quad (\text{A.8})$$

<sup>20</sup>Our notations corresponds to Waldschmidt's paper of ref. [54].

<sup>21</sup>We call  $n$  *depth*, and  $k = k_1 + k_2 + \dots + k_n$  ( $s = s_1 + s_2 + \dots + s_n$ ) the *weight*.

so that

$$\text{Li}_{\binom{\sigma_1, \sigma_2, \dots, \sigma_k}{s_1, s_2, \dots, s_k}}(y) = \int_0^1 \omega_0^{s_1-1} \omega_{\sigma_1} \omega_0^{s_2-1} \omega_{\sigma_1 \sigma_2} \cdots \omega_0^{s_k-1} \omega_{\sigma_1 \sigma_2 \cdots \sigma_k}, \quad \sigma_k^2 = 1. \quad (\text{A.9})$$

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